International Journal of Theoretical Physics, Vol. 15, No. 5 (1976), pp. 315-317

A Simple Application of the Newman-Penrose Spin Coefficient Formalism. I

TALMADGE M. DAVIS

Department of Physics and Astronomy, Clemson University, Clemson, South Carolina 29631[1]

Received: 9 *June* 1975

Abstract

As a simple application of the Newman-Penrose spin coefficient formalism, useful for beginners, we find the vacuum spherical symmetry (Schwarzschild) solution. The calculations also show that all spherically symmetric metrics are Petrov type *D*.

The spin coefficient formalism developed by Newman and Penrose (NP) (1962) has proved quite useful in finding solutions to the Einstein equations. The NP formalism is constructed around a tetrad of null vectors l_μ, n_μ, m_μ and $\bar{m}_\mu$; l_μ and n_μ are real, m_μ and $\bar{m}_\mu$ are complex, the bar denoting complex conjugate. Space-time quantities such as the Ricci tensor and Weyl tensor are contracted with the tetrad vectors to give the Φ_{ab} $(a, b = 0, 1, 2)$ and Ψ_c $(c = 0, 1, 2, 3, 4)$, respectively.

If one equates the Riemann tensor tetrad components in terms of the Weyl and Ricci tensor components with the tetrad covariant derivatives of the complex Ricci rotation coefficients, one obtains the 18 NP equations. These NP equations are written in terms of the 12 complex spin coefficients, which in turn involve derivatives of the tetrad vectors. The Einstein equations are incorporated into the NP equations through the substitution of the Ricci tensor as a function of the energy-momentum tensor.

The other equations to be solved in general are the Bianchi identities in tetrad form and the commutators of the tetrad derivatives, which when allowed to operate on the space-time coordinates yield the "metric" equations. Beginning with geometric requirements on the null tetrad vectors l_μ and n_μ (e.g., geodesic, shear-free), one obtains the metric by solving these NP and metric equations and identities, usually making use of the NP special coordinates.

[1] Dr. Davis' present address is Department of Math and Physics, Cumberland College, Williamsburg, Kentucky 40769.

As an instructive example in the use of the NP formalism we have solved the NP equations for the vacuum spherical symmetry solution. The resulting differential equations will, of course, yield the Schwarzschild solution.

The general spherically symmetric metric is given by (Landau and Lifshitz, 1971)

$$ds^2 = e^{2v}dt^2 - e^{2u}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{1}$$

where u and v are functions of r and t only. We will do our calculations in the orthonormal frame

$$\begin{aligned} \omega^1 &= e^u dr \\ \omega^2 &= r\, d\theta \\ \omega^3 &= r \sin\theta\, d\phi \\ \omega^4 &= e^v\, dt \end{aligned} \tag{2}$$

The metric is now simply the Minkowski metric, and a NP null tetrad can be written as

$$\begin{aligned} l_\mu &= (\delta_\mu{}^1 + \delta_\mu{}^4)/\sqrt{2} \\ n_\mu &= (-\delta_\mu{}^1 + \delta_\mu{}^4)/\sqrt{2} \\ m_\mu &= (\delta_\mu{}^2 + i\delta_\mu{}^3)/\sqrt{2} \\ \overline{m}_\mu &= (\delta_\mu{}^2 - i\delta_\mu{}^3)/\sqrt{2} \end{aligned} \tag{3}$$

The spin coefficients are

$$\kappa = \sigma = \tau = \nu = \lambda = \pi = 0 \tag{4}$$

$$\rho = \mu = e^{-u}/\sqrt{2}r \tag{5}$$

$$\alpha = -\beta = \operatorname{cotan}\theta/2\sqrt{2}r \tag{6}$$

$$\epsilon = (1/2\sqrt{2})(u_{,4}e^{-v} - v_{,1}e^{-u}) \tag{7}$$

$$\gamma = -(1/2\sqrt{2})(u_{,4}e^{-v} + v_{,1}e^{-u}) \tag{8}$$

where the comma denotes partial differentiation.

Substitution of these into the NP equations yields

$$D\rho = \rho^2 + 2\epsilon\rho + \Phi_{00} \tag{9}$$

$$\Delta\rho = -\rho^2 - 2\gamma\rho - \Phi_{22} \tag{10}$$

$$D\gamma - \Delta\epsilon = -4\epsilon\gamma + \Psi_2 - \Lambda + \Phi_{11} \tag{11}$$

$$\Phi_{11} = \Psi_2 - \Lambda - \rho^2 + r^{-2}/2 \tag{12}$$

$$\Phi_{00} = \Psi_2 + 2\Lambda - 4\epsilon\rho \tag{13}$$

$$\Phi_{22} = \Psi_2 + 2\Lambda - 4\gamma\rho \tag{14}$$

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \tag{15}$$

$$\Phi_{01} = \Phi_{02} = \Phi_{21} = 0 \tag{16}$$

Note that from (15) spherically symmetric metrics are Petrov type D (Newman and Penrose, 1962).

In empty space the Ricci tensor and its tetrad components vanish:

$$\Phi_{00} = \Phi_{11} = \Phi_{22} = \Lambda = 0 \tag{17}$$

Equations (9) and (10) yield

$$u_{,4} = 0 \tag{18}$$

$$(u + v)_{,1} = 0 \tag{19}$$

$$\epsilon = \gamma \tag{20}$$

Subtracting equations (12) and (13) with the use of (17) and integrating the resulting differential equations, we get

$$e^{2u} = (1 - r_0/r)^{-1} \tag{21}$$

where r_0 is a constant. Now, integrating (19) we get

$$e^{2v} = (1 - r_0/r) f^2(t) \tag{22}$$

where $f(t)$ is an arbitrary function of t. Since e^{2v} multiplies; dt^2 in the metric, we can perform coordinate transformation

$$dt' = f(t)\, dt \tag{23}$$

and dropping the prime

$$ds^2 = (1 - r_0/r)dt^2 - (1 - r_0/r)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta\, d\theta^2) \tag{24}$$

which is the Schwarzschild metric.

These results are, of course, very well known; however, calculations of this type provide good practice for anyone learning the NP spin coefficient formalism. This procedure also tells us that all spherically symmetric space-times are Petrov type D—a fact that is otherwise not obtained so easily. We have found no other simple applications such as this in the literature.

Acknowledgment

The author wishes to thank Dr. John R. Ray for his helpful criticism of this paper.

References

Newman, E. T., and Penrose, R. (1962). *Journal of Mathematical Physics*, 3, 566.

Landau, L. D., and Lifshitz, E. M. (1971). *The Classical Theory of Fields.* Pergamon, Oxford.